ON A METHOD OF OPTIMIZATION OF SERVO-SYSTEMS

(OB ODNOM SPOSOBE OPTIMIZATSII SLEDIASHCHIKH SISTEM)

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1. A servo-system is considered which is described by a linear differential equation $L_n(y) = f(t)$ of order n. In regard to the given external force $f(t)$ it is assumed only that it belongs to the class F of p-times differentiable functions such that $|f^{(p)}(t)| \leqslant M_p$. The restriction of boundedness in modulus may also be imposed on other derivatives of $f(t)$ and on the function $f(t)$ itself.

As an indicator of the quality of the servo we shall use the modulus (absolute value) of the difference $y(t) - f(t)$ on the interval $[0, T]$.

Sometimes it is possible to measure the values of the first k derivatives of the function $f(t)$.

It is assumed that the high-frequency noises and interferences have been filtered out when the function $f(t)$ enters the servo-system. In this case, in order to improve the quality of work of the servo, one can feed into the system, together with $f(t)$, also a linear combination

$$c_1(t) f'(t) + c_2(t) f''(t) + \ldots + c_k(t) f^{(k)}(t)$$

where the $c_i(t)$ belong to the class of A_i-functions. The classes of A_i-functions are determined by technical considerations.

Thus, the described system has the form

$$L_n(y) \equiv a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = f(t) + c_1(t) f'(t) + c_2(t) f''(t) + c_k(t) f^{(k)}(t) \tag{1.1}$$

$$f(t) \in F, \qquad c_i(t) \in A_i \ (i = 1, \ldots, k), \qquad t \in [0, T] \tag{1.2}$$

$$y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0, \qquad f(0) = f'(0) = \ldots f^{(p-1)}(0) = 0 \tag{1.3}$$

Its solution is

$$y(t) = Y(t, f(\tau), c(\tau)) \tag{1.4}$$

where $c(t)$ is a vector function, with coordinates $c_1(t), \ldots, c_k(t)$, and can be treated as a functional.

Let us now formulate the problem under consideration. It is required to find such functions $c_i(t)$ $(i = 1, \ldots, k)$ for which we have

$$E = \min_c \max_{t,f} |Y(t, f, c) - f(t)| \tag{1.5}$$

or

$$I = \min_c \max_f |Y(T, f, c) - f(T)| \tag{1.6}$$

Here t, $c(t)$, and $f(t)$ are chosen from (1.2); one deals here with absolute maxima and minima. In general I is less than E. Therefore, it is sometimes necessary to create a system which realizes (1.6) and not (1.5). We note that the measuring of the derivative functions of $f(t)$ involves technical difficulties, and these difficulties increase with the order of the derivatives. It is, therefore, desirable to obtain acceptable values of E or I with the aid of the smallest possible number of derivatives of $f(t)$. This can be accomplished to a certain extent through the enlargement of the classes A_i within, of course, certain technical limitations.

Similar problems arise in the creation of systems which are invariant relative to disturbances on a finite time interval, or at a fixed instant of time.

In the next sections we shall consider the problems stated above, together with certain other ones for the classes A_i and F.

2. In this section we shall assume that

$$c_i(t) = c_i, \quad |c_i| \leqslant m_i, \qquad (i = 1, \ldots, k), \qquad |f^{(p)}(t)| \leqslant M_p, \qquad t \in [0, T] \tag{2.1}$$

Taking into account the first equation of (1.3), we can express the solution of Equation (1.1) in the form

$$y(t) \equiv Y(t, f, c) = \int_0^t K(t-\tau)\Big(f(\tau) + \sum_{i=1}^k c_i f^{(i)}(\tau)\Big)\, d\tau \tag{2.2}$$

From the second group of equations in (1.3) we obtain

$$f^{(i)}(\tau) = \int_0^\tau f^{(p)}(u)\, \frac{(\tau - u)^{p-i-1}}{(p-i-1)!}\, du \qquad (i = 0, 1, \ldots, l), \quad l = \min(k, p-1) \tag{2.3}$$

Substituting (2.3) into (2.2), and changing the limits of integration in the repeated integrals obtained, we find

$$y(t) - f(t) = \int_0^t \Big[K_0(t-\tau) + \sum_{i=1}^k c_i K_i(t-\tau) \Big] f^{(p)}(\tau)\, d\tau$$

Here

$$K_0(t-\tau) = \int_\tau^t \frac{K(t-u)(u-\tau)^{p-1}}{(p-1)!}\, du - \frac{(t-\tau)^{p-1}}{(p-1)!}$$

$$K_i(t-\tau) = \int_\tau^t \frac{K(t-u)(u-\tau)^{p-i-1}}{(p-i-1)!}\, du \qquad (i = 1, \ldots, l)$$

If $k = p$, then $K_p(t-\tau) = K(t-\tau)$. Since $f^{(p)}(\tau)$ satisfies (2.1) we have [1]

$$A(c, t) = \max_f |Y(t, f, c) - f(t)| = M_p \int_0^t \Big| K_0(t-\tau) + \sum_{i=1}^k c_i K_i(t-\tau) \Big| d\tau \qquad (2.4)$$

It follows from this that $A(c, t_2) > A(c, t_1)$ when $t_2 > t_1$, and hence

$$A^*(c) = \max_t A(c, t) = M_p \int_0^T \Big| K_0(T-\tau) + \sum_{i=1}^k c_i K_i(T-\tau) \Big| d\tau$$

Thus, $E = I$ in the given case, and in order to find the $c_1, \ldots, c_k$ for which E is realized, one must minimize the expression

$$\int_0^T \Big| K(T-\tau) + \sum_{i=1}^k c_i K_i(T-\tau) \Big| d\tau \equiv \int_0^T |\varphi(\tau)|\, d\tau$$

The quantity E can be attained either at interior points of the k-dimensional parallelepiped V in the space with coordinates $c_1, \ldots, c_k$ determined by the relations (2.1), or at points belonging to its boundary. A necessary condition for the existence of a minimum inside the region V is given by the equations

$$\frac{\partial A^*}{\partial c_1} = \frac{\partial A^*}{\partial c_2} = \ldots = \frac{\partial A^*}{\partial c_k} = 0 \qquad (2.5)$$

One can show that

$$\frac{\partial A^*}{\partial c_i} = M_p \int_0^T K_i(T-\tau)\, \mathrm{sign} \Big[K_0(T-\tau) + \sum_{j=1}^k c_j K_j(T-\tau) \Big] d\tau \qquad (2.6)$$

Thus, in order that (1.5) may be realized at an interior point $c(c_1, \ldots, c_k)$ of the parallelepiped V, it is necessary that the following equations in the unknowns $c_1, \ldots, c_k$ be satisfied:

$$\int_0^T K_i\ (T-\tau)\ \mathrm{sign}\left[K_0(T-\tau)+\sum_{j=1}^{k} c_j K_j(T-\tau)\right] d\tau = 0 \qquad (i=1,\ldots,k) \quad (2.7)$$

If the point c is located on the boundary V_i of the parallelepiped V where the coordinates with the indices $i_1, \ldots, i_d$ take on their limiting values, then one has to cross out in the system (2.7) the equations with the corresponding indices, and in the remaining equations one must write in place of $c_{i_1}, \ldots, c_{i_d}$ their values on this boundary. The condition (1.5) may be attained on any one of the vertices of V. Since in the problem under consideration k cannot be large, the computation of 2^k values of $A^*(c)$ can be performed with the aid of a digital computing machine.

3. Let us consider the equation

$$L(y) = f(t) + c(t) f'(t), \quad |f'(t)| \leqslant M_1, \quad |c(t)| \leqslant m \qquad (3.1)$$

It is required to find $c(t)$ such that (1.6) may be realized. Representing the solution of Equation (3.1) in integral form, and making a few transformations similar to those performed in Section 2, we obtain

$$\mathrm{Y}(T, f, c) - f(T) = \int_0^T [K_0(T-\tau) + c(\tau) K(T-\tau)] f'(\tau)\, d\tau$$

$$K_0(T-\tau) = \int_\tau^T K(T-\tau)\, d\tau - 1$$

From this it follows immediately

$$A(c) = \max_f |\mathrm{Y}(T, f, c) - f(T)| = M_1 \int_0^T |K_0(T-\tau) + c(\tau) K(T-\tau)|\, d\tau$$

It is obvious that for the minimizing of $A(c)$ it is necessary and sufficient to minimize the value of the integrand function for any value of τ from $[0, T]$. Hence, the $c(t)$ for which I is realized must have the form

$$c(\tau) = -\frac{K_0(T-\tau)}{K(T-\tau)} \quad \text{for} \quad \left|\frac{K_0(T-\tau)}{K(T-\tau)}\right| \leqslant m$$

$$c(\tau) = -m\ \mathrm{sign}\frac{K_0(T-\tau)}{K(T-\tau)} \quad \text{for} \quad \left|\frac{K_0(T-\tau)}{K(T-\tau)}\right| > m$$

If the first inequality is satisfied for every $r \in [0, T]$, then $I = 0$, i.e. for all $f(t)$ satisfying (3.1), the difference $Y(T, f, c) - f(T) = 0$ at the instant of time T.

4. Let

$$L(y) = f(t) + \sum_{i=1}^{k} c_i(t) f^{(i)}(t), \qquad |f^{(p)}(t)| \leqslant M_p \tag{4.1}$$

$$c_i(t) = c_{ij} \text{ when } t \in [t_j, t_{j+1}), \qquad (j = 0, \ldots, r), \qquad t_0 = 0,\ t_{r+1} = T,\ |c_{ij}| \leqslant m_j \tag{4.2}$$

It is required to find $c_i(t)$ for which I is realized. We shall show that this problem can be reduced to the one considered in Section 2. Making use of (4.1) and (4.2), one can obtain

$$Y(T, f, c) - f(T) = \sum_{j=0}^{r} \left\{ \int_{t_j}^{t_{j+1}} K(T-\tau) \left[f(\tau) + \sum_{i=1}^{k} c_{ij} f^{(i)}(\tau) \right] - \frac{(T-\tau)^{p-1}}{(p-1)!} f^{(p)}(\tau)\, d\tau \right\}$$

We note that

$$\int_{t_j}^{t_{j+1}} K(T-\tau) f^{(i)}(\tau)\, d\tau = \int_0^T K_{ij}(u) f^{(p)}(u)\, du, \qquad (i = 1, \ldots, l), \qquad l = \min(k, p-1)$$

Here

$$K_{ij}(u) = \int_{t_j}^{t_{j+1}} \frac{K(T-\tau)(\tau-u)^{p-i-1}}{(p-i-1)!}\, d\tau \text{ when } u \in [0, t_j]$$

$$K_{ij}(u) = \int_{u}^{t_{j+1}} \frac{K(T-\tau)(\tau-u)^{p-i-1}}{(p-i-1)!}\, d\tau \text{ when } u \in (t_j, t_{j+1}]$$

$$K_{ij}(u) = 0 \text{ when } u \in (t_{j+1}, T]$$

Let us set

$$K_0(u) = \int_0^T \frac{K(T-\tau)(\tau-u)^{p-1}}{(p-1)!}\, d\tau - \frac{(T-u)^{p-1}}{(p-1)!}$$

and if $k = p$, then

$$K_{pj}(u) = K(T-u) \text{ when } u \subset [t_j, t_{j+1}]$$
$$K_{pj}(u) = 0 \text{ when } u \notin [0, T] \setminus [t_j, t_{j+1}]$$

Then

$$Y(T, f, c) - f(T) = \int_0^T \left[K_0(u) + \sum_{i=1}^{k} \sum_{j=0}^{r} c_{ij} K_{ij}(u) \right] f^{(p)}(u)\, du$$

$$A(c) = \max_f |Y(T, f, c) - f(T)| = M_p \int_0^T \left| K_0(u) + \sum_{i=1}^{k} \sum_{j=0}^{r} c_{ij} K_{ij}(u) \right| du$$

The problem is thus reduced to the one treated in Section 2. In the analyses of servo-systems there frequently arises the case when $k = p = 1$. In this case

$$A(c) = M_1 \int_0^T \left| K_0(u) + \sum_{j=0}^{r} c_{1j} K_{1j}(u) \right| du = M_1 \sum_{j=0}^{r} \int_{t_j}^{t_{j+1}} |K_0(u) + c_{1j} K(T-u)|\, du$$

Here the c_{1j} must be chosen so as to minimize

$$\int_{t_j}^{t_{j+1}} |K_0(u) + c_{1j} K(T-u)|\, du$$

If the interval $[t_j, t_{j+1}]$ is such that in it $K_0(u)$ and $K(T-u)$ are monotone, then this problem can be solved quite simply.

5. The actual evaluation of the derivatives of the function $f(t)$ is connected with considerable difficulties. Usually, when the differentiation is performed with the aid of electric systems, the output is a function $m(t)$ satisfying the equation

$$T_Э \frac{dm}{dt} + m = \frac{df}{dt} \tag{5.1}$$

For small values of the constant time $T_Э$ it is assumed that $m(t) \approx f'(t)$. From (5.1) it follows that

$$m(t) = \int_0^t \mu(t-\tau) f'(\tau)\, d\tau \tag{5.2}$$

In place of $f'(t)$ we obtain Expression (5.2) for the problem considered in Section 4. We have the equation

$$L(y) = f(t) + c(t) \int_0^t \mu(t-\tau) f'(\tau)\, d\tau$$

Thus

$$Y(T, f, c) - f(T) = \sum_{j=0}^{r} \int_{t_j}^{t_{j+1}} \left\{ K(T-\tau) \left[f(\tau) + c_j \int_0^T \mu(\tau-u) f'(u)\, du \right] - f'(\tau) \right\} d\tau$$

Interchanging the limits of integration of the repeated integrals, we obtain

$$\int_{t_j}^{t_{j+1}} K(T-\tau)\int_0^{\tau}\mu(\tau-u)f'(u)\,du\,d\tau=\int_0^T K_j(u)f'(u)\,du$$

Here

$$K_j(u)=\int_{t_j}^{t_{j+1}} K(T-\tau)\mu(\tau-u)\,d\tau \text{ when } u\in[0,t_j]$$

$$K_j(u)=\int_{u}^{t_{j+1}} K(T-\tau)\mu(\tau-u)\,d\tau \text{ when } u\in[t_j,t_{j+1}]$$

$$K_j(u)=0 \text{ when } u\in(t_{j+1},T]$$

Therefore

$$A(c)=\max_f|Y(T,f,c)-f(T)|=M_1\int_0^T\left|K_0(u)+\sum_{j=0}^{r}c_jK_j(u)\right|du$$

$$K_0(u)=\int_u^T K(T-u)\,du-1$$

From this it follows that one can state the problem on the minimization of $|Y(T, f, c) - f(T)|$ also with the aid of signals which reproduce the derivatives of $f(T)$ only approximately.

6. Let us consider the following problem. It is required to find a function $c(t)$ for which I is realized in the case that

$$L(y)=c(t)f(t),\qquad |c(t)|\leqslant M,\quad |f'(t)|\leqslant m_1,\qquad t\in[0,T]$$

Just as in the preceding sections, we write

$$|Y(T,f,c)-f(T)|=\left|\int_0^T[K_1(\tau)-1]f'(\tau)\,d\tau\right|,\qquad K_1(\tau)=\int_\tau^T K(T-u)c(u)\,du$$

Therefore

$$I=\min_c A(c)=\min_c m_1\int_0^T|K_1(\tau)-1|\,d\tau$$

We note that $K_1(T) = 0$ for any bounded function $c(u)$. From the form of the function $K_1(\tau)$ it follows at once that if $\min A(c)$ is to be attained, then it is necessary and sufficient that $|K_1(\tau) - 1|$ be as small as possible for any arbitrary value of τ from $[0, T]$. Hence, I

will be realized for the following function:

$c_0(u) = M \operatorname{sign} K(T-u)$ when $u \in [t_0, T]$; $c_0(u) = 0$ when $u \in [0, t_0]$, where t_0 is the root nearest to T of the equation for τ

$$M \int_{\tau}^{T} |K(T-u)|\, du = 1$$

We have considered above problems for the most simple, but also more important classes A_i and F. The problems on the determination of (1.5) for the problems of Sections 3 and 6 are more difficult. We call attention to the fact that no algorithm has been given for the solution of the system (2.7) even though the solution is relatively simple in many special cases.

7. As an example let us consider the equation

$$\ddot{y} + y = F(t)$$

As is known, for this equation $K(T-\tau) = \sin(T-\tau)$.

1. Let

$$F(t) = f'(t) + cf(t), \qquad |f'(t)| \leqslant m_1$$

In this case

$$A(c, t) = \max_f |Y(T, f, c) - f(T)| = m_1 \int_0^T |-\cos(T-\tau) + c\sin(T-\tau)|\, d\tau$$

$$A'(c, t) = \int_0^T \sin(T-\tau) \operatorname{sign}(-\cos(T-\tau) + c\sin(T-\tau))\, d\tau$$

Let $T = \pi/2$. It is not difficult to show that $A'(c, \pi/2) = 0$ only when $c = \sqrt{3}/3$, and that $A(c, \pi/2)$ has a minimum at this point

$$E = I = m_1 \int_0^{\frac{\pi}{2}} \left| -\cos\left(\frac{\pi}{2} - \tau\right) + \frac{\sqrt{3}}{3} \sin\left(\frac{\pi}{2} - \tau\right) d\tau = 0.73\, m_1\right.$$

If one does not introduce a signal proportional to $f'(t)$, i.e. if $c = 0$, then

$$\max_f \left| y\left(\frac{\pi}{2}\right) - f\left(\frac{\pi}{2}\right) \right| = m_1 \int_0^{\pi/2} |\cos\tau|\, d\tau = m_1$$

2. Let

$$F(t) = f(t) + c(t) f'(t)$$

where

$$c(t) = c_1 \quad \text{for} \quad t \in [0, t_1], \qquad c(t) = c_2 \quad \text{for} \quad t \in [t_1, T] \qquad |f'(t)| \leqslant m_1$$

From the relations obtained in Section 4, it follows that

$$A(c, t) = m_1 \left[\int_0^{t_1} |-\cos(T-\tau) + c_1 \sin(T-\tau)| \, d\tau + \right.$$

$$\left. + \int_{t_1}^{T} |-\cos(T-\tau) + c_2 \sin(T-\tau)| \, d\tau = A_1(c_1, T) + A_2(c_2, T) \right.$$

Let $t_1 = \pi/4$, $T = \pi/2$. In order to find c_1 and c_2 for which I is realized, we set the derivatives of the functions $A_1(c_1\pi/2)$ and $A_2(c_2\pi/2)$ equal to zero. It is not difficult to show that

$$A_1'\left(c_1, \frac{\pi}{2}\right) = 0 \quad \text{for} \quad c_1 = 0.377, \quad A_2'\left(c_2, \frac{\pi}{2}\right) = 0 \quad \text{for} \quad c_2 = 1.64$$

These are the only extremal values of these functions. At them I is realized, and $I = 0.49\ m_1$.

Note 7.1. We call attention to the fact that all the above-considered methods for finding I apply also to the case when $L(y)$ has variable coefficients.

Note 7.2. The results obtained can be generalized to the case when $L(y)$ is a linear difference operator. This can be done with a transformation given in [2].

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